Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

J. Math. Anal. Appl. 334 (2007) 414–430

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On stability of switched homogeneous nonlinear systems ☆

Lijun Zhang *, Sheng Liu, Hai Lan

School of Automation, Harbin Engineering University, Harbin 150001, China

Received 27 May 2006

Available online 4 January 2007

Submitted by Z.S. Athanassov

Abstract

In this paper, the problem of stability of switched homogeneous systems is addressed. First of all, if there is a quadratic Lyapunov function such that nonlinear homogeneous systems are asymptotically stable, a matrix Lyapunov-like equation is obtained for a stable nonlinear homogeneous system using semi-tensor product of matrices, and Lyapunov equation of linear system is just its particular case. Following the previous results, a sufficient condition is obtained for stability of switched nonlinear homogeneous systems, and a switching law is designed by partition of state space. In particular, a constructive approach is provided to avoid chattering phenomena which is caused by the switching rule. Then for planar switched homogeneous systems, an LMI approach to stability of planar switched homogeneous systems is presented. Similar to the condition for linear systems, the LMI-type condition is easily verifiable. An example is given to illustrate that candidate common Lyapunov function is a key point for design of switching law.

Keywords: Switched homogeneous nonlinear systems; Stability; Semi-tensor product of matrices; LMI approach

1. Introduction

In recent years, a study of switched systems has received more and more attention. Many engineering systems, such as robot manipulators, traffic management, power systems, etc. are essentially switched systems. Study of switched systems mainly focuses on stability, stabilization,

☆ Work supported partly by the NNSF 60334040 of China and Fundamental Fund of Harbin Engineering University.
* Corresponding author. Fax: +86 451 82519400.
E-mail addresses: zhanglj@hrbedu.edu.cn (L. Zhang), liusheng@hrbeu.edu.cn (S. Liu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.12.065

controllability, and optimal control, etc. Many interesting results can be found in various relevant literatures [2,4,5,1,6,7]. In particular, research of switched nonlinear systems is developed rapidly in the last few years. Some results for stability of switched nonlinear system are found in [24,11,13]. Other syntheses, such as controllability and disturbance decoupling of switched nonlinear system, have been investigated in [3,23]. Various methods, such as common Lyapunov function, LMI approach and nonlinear programming, etc. have been widely used to address different problems of switched systems.

The problems of stability or stabilization of switched systems include two aspects: one is how to make switched systems stable or stabilized under arbitrary switching law, where each switching subsystem is required to be stable. Common Lyapunov function is a powerful tool to tackle this kind of problems. Lie algebraic approach is another important tool. The other is how to design a switching law under which switched systems are stable or stabilized, where each switching subsystem is unstable or unstabilizable. Some methods such as multi-Lyapunov functions and convex combination of vector fields have been used widely. For the first case, there are too many lectures to investigate stability for switched linear systems, here we only recall some available results of stability for nonlinear systems. Reference [8] has given the commute matrix condition of fields for stability of nonlinear systems. Reference [9] has relaxed the condition of [8] to some nilpotent Lie algebras using an optimal control idea. References [12] and [13] have given ISS and exponential stability of switched nonlinear systems when switching dwell time is enough long. The existence of converse Lyapunov function of switched nonlinear systems is also an interesting problem, some efforts have been made to investigate this kind of problems [10,24]. Compared with the first case, the second case is a kind of more interesting and challenging problem just since a suitable switching law can drive trajectory of switched systems with unstable switching subsystems to equilibrium along the designed switching path. However, there is not a suitable tool to design stabilizing switching law yet. In addition, chattering phenomena caused by switching critical condition is boring, which may lead to fatal damages for practical engineering operator. The first lecture involving design of switching law is [14], where stability of switched linear systems has been considered. If a convex hull of unstable matrix is stable, so are switched systems. Later on, many lectures focus on analysis and control design of the systems using this approach [16]. Reference [16] has provided the specific switching path based on the result of [14]. We has also presented a switching law by geometric method in [22]. However, for switched nonlinear systems, nonlinearity of systems leads to tremendous difficulties in designing some computable switching law. To our best of knowledge, few lectures involving switching law design of nonlinear systems are found. To provide practical and easily verified stability methods for switched nonlinear systems, it is necessary to avoid the general nonlinear systems, specific systems have to be considered.

As is well known, the systems with homogeneous properties have special characteristics similar to those of linear systems, therefore better theoretic results can often be obtained than general affine nonlinear systems. There are some good studies of homogeneous systems in different aspects [19–21]. Global stabilization of homogeneous systems has been proved by geometric inequality method in [21], and [19] has given a sufficient condition of H_∞ control of homogeneous systems similar to that of linear systems. Of course, some results of switched homogeneous systems also appear in some lectures. Reference [17] has analyzed stability of second-order switched homogeneous systems by generalized first integrals. Smooth Lyapunov function for homogeneous differential inclusions has been obtained in [18], where the existence of a smooth Lyapunov function has been proved. However, [17,18] has only considered stability of such switched systems with all stable subsystems. Switched homogeneous systems with unstable subsystems have been not covered. In addition, only the stability has been studied rather than the quadratic stability of switched homogeneous systems which is just what our paper focuses on.

This paper focuses on the following switched nonlinear systems

$$\dot{x} = f_{\sigma(x,t)}(x), \quad x(t) \in \mathbb{R}^n, \tag{1}$$

where the switching law $\sigma(t,x):([0,\infty),\mathbb{R}^n) \to \Lambda$ is a right-continuous piecewise constant mapping and $\Lambda = \{1, 2, \ldots, N\}$ for some integer $N \geqslant 2$, $f_{\sigma(t,x)}$ are homogeneous vector fields of odd degree k.

In this paper we use a kind of new matrix product, semi-tensor product of matrices, to solve the problem of stability of homogeneous systems. Semi-tensor product of matrices is a kind of new multiple operator of matrices, it can solve some problems of polynomials for which conventional matrix product cannot work. Semi-tensor product of matrices has been used for some fields of control theory such as input–output decoupling, attractive region, etc. [25,26]. Especially, a vector polynomial can be denoted as linear form by semi-tensor product of matrices. Using semi-tensor product of matrices, we obtain Lyapunov-like equation of stable homogeneous systems, which has the same form as that of linear systems. Therefore, for some kind of homogeneous systems, we can use matrix methods to study their stability and other properties as in linear systems. Furthermore, we can also investigate stability of switched homogeneous systems by LMI approach just as in linear systems. As a result, the condition for stability obtained is easily verified by computer. The main contributions of this paper are listed as follows:

(1) Using semi-tensor product of matrices, a Lyapunov-like equation similar to that of linear systems is presented. The Lyapunov equation of linear systems is its special case.
(2) Based on the result of (1), a switching law is designed to stabilize switched homogeneous nonlinear systems (1). Chattering phenomena caused by switching rule is avoided by topological techniques.
(3) For planar switched homogeneous system, a easily verified sufficient condition for stability is obtained by LMI approach.
(4) In addition, some results with respect to semi-tensor product of matrices are meaningful themselves. They further develop this theory.

The rest of this paper is organized as follows. Based on the quadratic Lyapunov function of homogeneous systems and semi-tensor product of matrices, a Lyapunov-like equation is given in Section 2. In Section 3, a sufficient condition is obtained to ensure system (1) stable, and a switching law is designed to stabilize the system and chattering phenomena caused by the switching rule can be avoided by topological techniques. A sufficient condition by LMI approach is obtained for the stability of planar switched homogeneous systems in Section 4. An illustrating example is given in Section 5, and followed by Section 6 which concludes the work. Finally, the preliminaries of semi-tensor product of matrices are given in Appendix A.

2. Lyapunov-like equation of homogeneous nonlinear system

Consider the following homogeneous system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $f(x)$ is a homogeneous polynomial vector field of odd degree k.

The following lemma is important for our immediate discussion.

Lemma 1. *(See [29].) If system* (2) *is asymptotically stable, then there is a positive matrix P such that its Lyapunov function $V(x) = x^T Px$.*

Remark 1. According to Lemma 1, the existence of the quadratic Lyapunov function is a sufficient and necessary condition for stable homogeneous systems.

Next, we will use semi-tensor product of matrices (see Appendix A) to show Lyapunov-like equation of stable homogeneous system. First we give semi-tensor product representation of a homogeneous polynomial.

Proposition 10 (see Appendix A) implies that a homogeneous polynomial of degree k can be expressed as semi-tensor product of matrices, i.e.,

$$f(x) = A \ltimes x^k, \quad A \in M_{1\times n^k}.$$

The following example gives a specific description.

Example 1. Given the homogeneous polynomial

$$f(x) = x_1^3 + x_1^2 x_2 - 2x_1 x_2^2 - x_2^3. \tag{3}$$

Note that

$$x = [x_1 \quad x_2]^T,$$
$$x^3 = x \ltimes x \ltimes x = \begin{bmatrix} x_1^3 & x_1^2 x_2 & x_1 x_2 x_1 & x_1 x_2^2 & x_2 x_1^2 & x_2 x_1 x_2 & x_2^2 x_1 & x_2^3 \end{bmatrix}^T.$$

Hence $f(x)$ can be expressed as

$$f(x) = [1 \quad 1 \quad 0 \quad -2 \quad 0 \quad 0 \quad 0 \quad -1] \ltimes x^3,$$

where $A = [1\ 1\ 0\ {-2}\ 0\ 0\ 0\ {-1}]$ is a 1×8 matrix.

Since x^k is a redundant basis, the expression of A is not unique. But matrix A is expressed uniquely as in [27, p. 82]. Then matrix A in Example 1 is rewritten as

$$A = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & -1 \end{bmatrix}.$$

Note that unique quality of matrix A is a foundation of our further discussion.

The following lemma is helpful for the immediate main result.

Lemma 2. *Assume M is an $n^s \times n^s$ matrix, and x is a column vector of dimension n, then $V_r^T(M)x^{2s} = (x^s)^T M x^s$.*

Proof.

$$V_r^T(M)x^{2s} = V_r^T(M) \ltimes x^s \ltimes x^s = \left(V_r^T(M) \ltimes x^s\right) \ltimes x^s = \left(x^T\right)^s M x^s.$$

The last equality holds by Proposition 10 (see Appendix A). $\square$

According to Lemma 1, there exists a quadratic Lyapunov function when system (2) is asymptotically stable. Let $V(x) = x^T Px$ is the Lyapunov function.

Now system (2) can be expressed as

$$\dot{x} = Ax^k, \quad x \in \mathbb{R}^n, \tag{4}$$

where A is an $n \times n^k$ matrix.

The following theorem is the first main result of this paper.

Theorem 1. *When system* (4) *is asymptotically stable, there is a Lyapunov-like equation similar to that of linear system.*

Proof. Note

$$x^T PAx^k = \left(x^T PA\right)x^k = V_r^T(PA)x^{k+1}.$$

The last equality holds by Propositions 5 and 10 (see Appendix A).

Let $k+1 = 2s$, where s is a positive integer. Since $V_r^T(PA)$ is a $1 \times n^{k+1}$ matrix, i.e. it has $n^{k+1} = (n^s)^2$ elements. So $V_r^T(PA)$ can be expressed as a $n^s \times n^s$ square matrix, the reason is as follows:

Since for any matrix $M_{m\times n}$, we have

$$M = \left[I_m \otimes V_r^T(I_n)\right] \ltimes V_r(M).$$

Then we have

$$\left[I_{n^s} \otimes V_r^T(I_{n^s})\right] \ltimes V_r(PA)$$

is an $n^s \times n^s$ square matrix, i.e., $V_r(PA)$ can be converted as a square matrix.

Based on Lemma 2, we have

$$V_r^T(PA)x^{k+1} = \left(x^s\right)^T V_r\left[I_{n^s} \otimes V_r^T(I_{n^s})\right] \ltimes V_r(PA)x^s. \tag{5}$$

By Proposition 9 (see Appendix A), we have

$$V_r(PA) = P \ltimes V_r(A). \tag{6}$$

Then by (6), (5) can be rewritten as

$$V_r^T(PA)x^{k+1} = \left(x^s\right)^T \left[I_{n^s} \otimes V_r^T(I_{n^s})\right] \ltimes P \ltimes V_r(A)x^s. \tag{7}$$

By Proposition 6 (see Appendix A), we have

$$V_r^T(PA)x^{k+1} = \left(x^s\right)^T (P \otimes I_{n^{s-1}})\left[I_{n^s} \otimes V_r^T(I_{n^s})\right] \ltimes V_r(A)x^s. \tag{8}$$

Note that

$$\left[I_{n^s} \otimes V_r^T(I_{n^s})\right] \ltimes V_r(A)$$

just is the square form of matrix A.

Similar to the case of linear system, we take the symmetric form of $\dot{V}$:

$$\begin{aligned}\dot{V} = \left(x^s\right)^T \Big(\big[\left[I_{n^s} \otimes V_r^T(I_{n^s})\right] \ltimes V_r(A)\big]^T (P \otimes I_{n^{s-1}}) \\ + (P \otimes I_{n^{s-1}})\left[I_{n^s} \otimes V_r^T(I_{n^s})\right] \ltimes V_r(A)\Big)x^s.\end{aligned} \tag{9}$$

Now let

$$P \otimes I_{n^{s-1}} = \tilde{P}, \tag{10}$$

$$\left[I_{n^s} \otimes V_r^T(I_{n^s})\right] \ltimes V_r(A) = \tilde{A}, \tag{11}$$

then (9) can be expressed as

$$\dot{V} = \left(x^s\right)^T \left(\tilde{P}\tilde{A} + \tilde{A}^T \tilde{P}\right)x^s, \tag{12}$$

where $\tilde{P}$ is still a positive matrix. $\square$

Note (12) is just Lyapunov equation similar to that of linear system, and when the degree $k = 1$, i.e., system (4) is linear, it is just the well-known Lyapunov equation. So does the following corollary.

Remark 2. It is clear that $\tilde{P}$ is a positive matrix based on eigenvalues of tensor product of matrices. $\tilde{A}$ is only a square matrix transformed by matrix A.

Corollary 1. *For linear system $\dot{x} = Ax$, if $V = x^T Px$ is its Lyapunov function, then* (12) *can be reduced to*

$$\dot{V} = x^T\left(PA + A^T P\right)x.$$

Proof. When $k = 1$, $s = 1$, we have

$$\left[I_{n^s} \otimes V_r^T(I_{n^s})\right] \ltimes P = \left[I_n \otimes V_r^T(I_n)\right] \ltimes P.$$

A straightforward compute yields

$$\left[I_n \otimes V_r^T(I_n)\right] \ltimes P = P\left[I_n \otimes V_r^T(I_n)\right]. \tag{13}$$

By (13), the right-hand side of (8) can be rewritten as

$$x^T\left(P\left[I_n \otimes V_r^T(I_n)\right] \ltimes V_r^T(A)\right)x.$$

Since $[I_n \otimes V_r^T(I_n)] \ltimes V_r^T(A) = A$, we have

$$\dot{V} = x^T\left(PA + A^T P\right)x. \qquad \square$$

3. Stability of switched homogeneous nonlinear systems

In this section, the stability of switched system (1) is studied. Here we only consider $\sigma(x,t) = \sigma(x)$, i.e. state feedback switching law. A switching law is designed to stabilize the whole system.

Note system (1) can be expressed as

$$\dot{x} = f_i(x), \quad x \in \mathbb{R}^n,\ i \in \Lambda. \tag{14}$$

Further

$$\dot{x} = A_i x^k, \quad x \in \mathbb{R}^n,\ i \in \Lambda. \tag{15}$$

To begin with, we give the definition for the quadratic stabilization.

Definition 1. Consider a dynamic system

$$\dot{x} = Ax^k, \quad x \in \mathbb{R}^n. \tag{16}$$

1. (16) is said to be quadratically asymptotically stable if there exists a positive definite matrix P such that
$$\left(x^s\right)^T\left(\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}\right)x^s<0, \tag{17}$$
where $\tilde{P}$ and $\tilde{A}$ is taken as in (12).
2. For a given positive definite matrix P, the stable region S_P is defined as
$$S_P(A)=\{0\}\cup\left\{x\in\mathbb{R}^n\;\middle|\;\left(x^s\right)^T\left(\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}\right)x^s<0\right\},$$
where $s=(k+1)/2$.

When the stable region is considered, the topology of projective space $P^{n-1}(\mathbb{R})$ provides a suitable structure for it. Because if $x\in S_P(A)$, then for any real number $\lambda\neq 0$, $y=\lambda x\in S_P(A)$ since system (14) is homogeneous. Hence, we simply identify them as $y\sim x$. Under this equivalent relation, the quotient space is $P^{n-1}(\mathbb{R})$, i.e.,
$$P^{n-1}(\mathbb{R})=\left(\mathbb{R}^n\setminus\{0\}\right)/\sim.$$
By Definition 1, it is clear that system (14) is asymptotically stable if there exists $P>0$, $S_P(A)=\mathbb{R}^n$.

For convenience, we technically remove zero from $S_P(A)$. That is, set
$$S_P(A)=\left\{x\in\mathbb{R}^n\;\middle|\;\left(x^s\right)^T\left(\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}\right)x^s<0\right\}.$$
Note our discussion can be restricted in unit sphere $S^{n-1}=\{x\in\mathbb{R}^n\mid\|x\|=1\}$. It is easy to see that
$$P^{n-1}(\mathbb{R})\backsimeq S^{n-1},$$
where $\backsimeq$ denotes that S^{n-1} is homeomorphic to $P^{n-1}(\mathbb{R})$ in a natural way. This is also why we can simply consider that the points x are on the sphere S^{n-1}.

The following result is an immediate consequence of Definition 1.

Proposition 1. *Let A_i, $i\in\Lambda:=\{1,\dots,N\}$, be a finite set of matrices. The switched system* (14) *is quadratically stabilizable by a state feedback switching law if there exists $P>0$ such that*
$$\bigcup_{i\in\Lambda}S_P(A_i)\cap S^{n-1}=S^{n-1}. \tag{18}$$

Proof. For any $x\in\bigcup_{i\in\Lambda}S_P(A_i)\cap S^{n-1}$, we can choose the quadratic Lyapunov function, $L(x)=x^TPx$. The state feedback switching law can be chosen as
$$\sigma(x)=\arg\min_i\left\{x\;\middle|\;\left(x^s\right)^T\left(\tilde{P}\tilde{A}_i+\tilde{A}_i^T\tilde{P}\right)x^s<0\right\}. \tag{19}$$

Then it is easy to see that under such a switching law $\dot{L}(x)$ is a continuous function. Note that since the system is not continuous we still have to show that the system is asymptotically stable. Given any $\epsilon>0$ consider
$$R=\left\{x\;\middle|\;\epsilon\leqslant L(x)\leqslant L(x_0)\right\}.$$
R is a compact set, which is invariant with respect to (14). By continuity of $\dot{L}$, $\dot{L}$ can reach its maximum value $\delta<0$. That is,
$$\dot{L}(x)\leqslant\delta<0,\quad x\in R.$$
Therefore, after a certain finite time $T>0$ we have $x(t)\in R_\epsilon:=\{x\mid L(x)<\epsilon\}$. □

For a switched system a serious problem is the vibration. For instance, the switching law maybe as: $\sigma(t_-) = i$ and $\sigma(t_+) = j$, $j \neq i$, and vise versa. That is, the system will go back and forth between these two models with 0_+ dwell time. If this kind of vibration occurs, even the existence of the solution is questionable. To avoid this, we have to modify the switching law (19). In the following we will design a new switching law, which will avoid this kind of vibration.

The following discussion is in S^{n-1} otherwise explanation.

Consider S^{n-1}, it is a compact topological space. Let

$$U_i := \{S_P(A_i)\}, \quad i \in \Lambda.$$

Then $\{U_i \mid i \in \Lambda\}$ form an open covering of S^{n-1}. Since P^{n-1} is a normal topological space [28], there exist open sets $V_i \subset \bar{V}_i \subset U_i$ such that

$$\bigcup_{i \in \Lambda} V_i \supset S^{n-1}.$$

Since $\bar{V}_i \subset U_i$, we have

$$(x^s)^T (\tilde{P}\tilde{A}_i + \tilde{A}_i^T \tilde{P}) x^s < 0, \quad x \in \bar{V}_i, \ i \in \Lambda.$$

Note that $\bar{V}_i$ is compact, we can find $\epsilon_i < 0$ such that

$$\max_{x \in \bar{V}_i} (x^s)^T (\tilde{P}\tilde{A}_i + \tilde{A}_i^T \tilde{P}) x^s < \epsilon_i < 0, \quad i \in \Lambda.$$

Now we modify the switching law of (19) as follows:

$$\sigma(x, t_+) = \arg\min \left\{ (x^s)^T (\tilde{P}\tilde{A}_j + \tilde{A}_j^T \tilde{P}) x^s, \ j \neq i; \right.$$
$$\left. (x^s)^T (\tilde{P}\tilde{A}_i + \tilde{A}_i^T \tilde{P}) x^s - \frac{\epsilon_i}{2} \right\}, \tag{20}$$

where i is the current model, i.e., $\sigma(x, t) = i$.

Note that under the switching law (20) if $\sigma(t_k)$ is a newly chosen model, then the system will stay in this model for a considerable time period to "consume" its $\frac{\epsilon_i}{2}$ "privilege." To see this, say at a moment t_0 we have

$$(x^s)^T (t_0) (\tilde{P}\tilde{A}_i + \tilde{A}_i^T \tilde{P}) x^s(t_0) = (x^s)^T (t_0) (\tilde{P}\tilde{A}_j + \tilde{A}_j^T \tilde{P}) x^s(t_0),$$

and $\sigma(x, t_0) = i$. Then the system will remain in model i until another moment t_1 when

$$(x^s)^T (t_1) (\tilde{P}\tilde{A}_j + \tilde{A}_j^T \tilde{P}) x^s(t_1) = (x^s)^T (t_1) (\tilde{P}\tilde{A}_i + \tilde{A}_i^T \tilde{P}) x^s(t_1) - \frac{\epsilon_i}{2},$$

and

$$(x^s)^T ((t_1)_+) (\tilde{P}\tilde{A}_j + \tilde{A}_j^T \tilde{P}) x^s((t_1)_+) < (x^s)^T ((t_1)_+) (\tilde{P}\tilde{A}_i + \tilde{A}_i^T \tilde{P}) x((t_1)_+) - \frac{\epsilon_i}{2}.$$

This delay in switching avoids vibration.

To see that the switched system is still quadratically asymptotically stable, it is because that

$$\dot{V} \leqslant \max_i \left\{ \frac{\epsilon_i}{2} \right\} < 0. \tag{21}$$

In fact, Proposition 1 is equivalent to the result presented in [16] which is an extension of [15]. The following proposition implies this fact.

Proposition 2. *For switched system* (14)*, if there exists a common Lyapunov function* $V(x) = x^T Px$ *such that* $\bigcup_{i\in\Lambda} S_P(A_i) \cap S^{n-1} = S^{n-1}$*, then there exist* $0 \leqslant \alpha_i \leqslant 1$ *and* $\sum_1^N \alpha_i = 1$ *such that for such a Lyapunov function the following inequality holds. So is inversion*

$$\Delta V(x) \sum_{i=1}^{N} \alpha^i A_i x^k < 0, \quad i \in \Lambda,\ \forall x \in S^{n-1}. \tag{22}$$

Proof. Let $\Omega_i = \{x \in S^{n-1} \mid \Delta V(x) < 0\}$. By (18), we have $S^{n-1} = \bigcap_{i=1}^N \Omega_i$, i.e. over unit sphere S^{n-1}, we have

$$\left(\bigcap_{i=1}^{N} \Omega_i\right)^c = \bigcup_{1}^{N} \Omega_i^c = \Phi$$

holds, where $(\cdot)^c$ denotes complement set of a set.

Further it means that solutions of the following set of inequalities are null in S^{n-1}

$$\begin{cases} \Delta V A_1 x^k \geqslant 0, \\ \vdots \\ \Delta V A_N x^k \geqslant 0. \end{cases} \tag{23}$$

Now we claim no solutions of (23) if there exist $0 \leqslant \alpha_i \leqslant 1$ and $\sum_{i=1}^N \alpha_i = 1$ such that (22) holds. We seek to a contradiction. If (23) is not true. There exists a $x_0 \in S^{n-1}$ such that for any α_i we have $\sum_{i=1}^N \alpha_i A_i x_0 \geqslant 0$ which contradicts (22). So is inversion. □

4. LMI approach to stability of switched homogeneous systems

LMI approach has been used for investigating stability and stabilization of switched linear systems [30] and references therein. However, LMI seems invalid in studies of stability of nonlinear systems for presence of nonlinear items, therefore, to my best knowledge, there are few lectures found to investigate stability of nonlinear systems by LMI approach. In this section, we present a sufficient condition for stability of system (14) by LMI approach.

Definition 2. A symmetric matrix P of dimension $n \times n$ is said to be s degree homogeneous positive (s-DHP) if the following inequality holds

$$\left(x^s\right)^T P x^s > 0, \quad \forall x \in \mathbb{R}^r \setminus \{0\}, \tag{24}$$

where $x^s = \underbrace{x \ltimes x \ltimes \cdots \ltimes x}_{s}$ and $sr = n$.

Definition 3. A symmetric matrix P of dimension $n \times n$ is r degree semi-tensor product positive (r-SPP) if the following inequality holds

$$x^T \ltimes P \ltimes x > 0, \quad x \in \mathbb{R}^r \setminus \{0\}, \tag{25}$$

where r is an integer factor of n.

Similarly, s degree homogeneous negative matrix (s-DHN) and r degree semi-tensor product negative matrix (r-SPN) can also be defined.

Lemma 3. *A positive (negative) symmetric matrix P is r-SPP (r-SPN).*

Proof. Without loss of generality, assume an $n \times n$ matrix P is positive, for any $x \in \mathbb{R}^r$ we have

$$x^T \ltimes P \ltimes x = \left(x^T \ltimes P\right) \ltimes x = \left(x^T \otimes I_s\right) P (x \otimes I_s) > 0,$$

where $rs = n$ and I_s is an identity matrix. The last inequality holds since P is positive. □

Remark 3. By Lemma 3 and Definition 3, we have $(x^s)^T P x^s > 0$, $s \leqslant k$, where P is an $n \times n$ symmetric matrix and $k = n/2$, conversion is not true.

By Definition 3, clearly we have the following proposition.

Proposition 3. *If a matrix P is r-SPP, for any $r \times r$ invertible matrix M, $M^T \ltimes P \ltimes M$ is also r-SPP.*

Next we give LMI approach to stability of switched homogeneous systems, here we only focus on $n = 2$, $s = 2$ case.

For any homogeneous polynomial of 4 degree $P(x) = (x^2)^T P x^2$, where $x \in \mathbb{R}^2$, P is a symmetric matrix. It is denoted as follows

$$P = \begin{bmatrix} p_{11} & p_{12} & 0 & p_{14} \\ p_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{34} \\ P_{41} & 0 & p_{43} & p_{44} \end{bmatrix}. \tag{26}$$

Clearly the above matrix can denote any homogeneous polynomial uniquely. Now assume $P(x) < 0$, it is easy to see $p_{ii} < 0$, $i = 1, 4$.

First we recall some results in [31], where Cross Row Diagonal Dominating Principle (CRDDP), Diagonal Dominating Principle (DDP) and Quadratic Form Reducing Algorithm (QFRA) were given to investigate stability of homogeneous polynomial of degree k. Details can refer readers to [31]. Reference [31] also proved that QFRA is stronger than CRDDP and CRDDP is stronger than DDP. In the following, we show that QFRA can be denoted as LMI form.

Lemma 4. *QFRA condition of homogeneous polynomial $P(x)$ is equivalent to the following Linear Matrix Inequality*

$$\begin{bmatrix} p_{11} + p_{12} & p_{14} + \frac{p_{12}+p_{34}}{2} \\ p_{41} + \frac{p_{21}+p_{34}}{2} & p_{44} + p_{34} \end{bmatrix} < 0. \tag{27}$$

Proof. It is a straightforward result of QFRA. □

Lemma 4 implies when matrix (27) is negative system, homogeneous polynomial $P(x)$ is also negative.

Next we will show main result of this section.

By Proposition 2, if inequality (22) holds, switched homogeneous system (15) is stabilizable via some designed switching law.

Assume Lyapunov function of system (15) is $V(x) = x^T P x$, by (12), inequality (22) can be rewritten as

$$\dot{V}(x) = \left(x^s\right)^T \left(P \ltimes \sum_{i=1}^{N} \alpha_i \tilde{A}_i + \sum_{i=1}^{N} \alpha_i \tilde{A}_i^T \ltimes P\right) x^s. \tag{28}$$

Let $\sum_{i=1}^{N} \alpha_i \tilde{A}_i = \bar{A}$, then (28) can be rewritten as

$$\dot{V}(x) = \left(x^s\right)^T \left(P \ltimes \bar{A} + \bar{A}^T \ltimes P\right) x^s. \tag{29}$$

Maybe $P \ltimes \bar{A} + \bar{A}^T \ltimes P$ is not of (26) form. But this will not cause any difficulty since it is equivalent to (26) in k-DHP sense. As such, we may first denote $P \ltimes \bar{A} + \bar{A}^T \ltimes P$ by form of (26), and then using (27), negativity of (29) is easy to be verified. We call a 4×4 symmetric matrix M denoted by (26) form as normalization. Denoted by $\mathfrak{N}(M)$.

Consequently, we have the following theorem.

Theorem 2. *Planar system* (15) *of degree* 3 *is asymptotically stabilizable via designed switching law if LMI*

$$\mathfrak{N}\left(P \ltimes \bar{A} + \bar{A}^T \ltimes P\right) < 0$$

holds.

Remark 4. In fact, Theorem 2 can be extended to planar general homogeneous systems of k odd degree by similar method.

Now a question have to be considered: does common Lyapunov function exist? Is it easy to find? The following theorem implies Lyapunov function is no more difficult for switched homogeneous systems to find than that of switched linear systems.

Theorem 3. *Assume system* (15) *is stable, for any s-DHP matrix Q, there exists a positive matrix P such that*

$$\left(x^s\right)^T \left(P \ltimes \tilde{A} + \tilde{A}^T \ltimes P\right) x^s = -\left(x^s\right)^T Q x^s. \tag{30}$$

Proof. Assume Lyapunov function

$$V(x) = \int_0^\infty \left(\psi(\tau, x, 0)^s\right)^T Q \psi(\tau, x, 0)^s \, d\tau \tag{31}$$

$$= \int_t^\infty \left(\psi(\tau, x, t)^s\right)^T Q \psi(\tau, x, 0)^s \, d\tau, \tag{32}$$

where $\psi(t, x, 0)$ is a solution of system (2) with initial state $x = \psi(0, x, 0)$ and $2s = k + 1$. It is easy to see $V(x_0) = V(x(t)) + \int_0^t (\psi(\tau, x, t)^s)^T Q \psi(\tau, x, 0)^s \, d\tau$. Then we have

$$\dot{V}|_2 = -\left(x^s\right)^T Q x^s.$$

Similar to proof of Lemma 1 of [29], it is easy to show $V(x)$ is still a homogeneous function with degree 2. Consequently positive definite matrix P exists such that $V(x) = x^T P x$. $\square$

Remark 5. When $Ax^k = (x^l)^T B x^l C x$, where $(x^l)^T B x^l > 0$ and C is Hurwitz stable. Then Lyapunov positive matrix P of system (15) can be taken the same as that of system $\dot{x} = Cx$.

It is well known the choice of matrix $P > 0$ is a key for switching law. So far, there are some algorithm methods to find numerical solution of Lyapunov function for linear case. However, it is very difficult to get Lyapunov function just for nonlinearity.

5. Illustrating example

Consider switched system (14) with two switching models A_1 and A_2 as follows. The following example describes different Lyapunov function can generate different stable region of two matrices.

Consider the following switched system with two switching models.

Example 2. Consider two matrices

$$\begin{cases} \dot{x}_1 = x_1^3 + 2x_1x_2^2 + 2x_1^2x_2 + 4x_2^3, \\ \dot{x}_2 = -x_1^3 - 2x_1x_2^2 - 3x_1^2x_2 - 6x_2^3, \end{cases} \tag{33}$$

$$\begin{cases} \dot{x}_1 = -2x_1^3 - 4x_1x_2^2 + x_1^2x_2 + 2x_2^3, \\ \dot{x}_2 = -0.5x_1^3 - x_1x_2^2 + 3x_1^2x_2 + 6x_2^3, \end{cases} \tag{34}$$

or written as

$$\dot{x} = A_1x^3, \tag{35}$$

$$\dot{x} = A_2x^3, \tag{36}$$

where $x^3 = x \ltimes x \ltimes x$, and

$$A_1 = \begin{bmatrix} 1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 4 \\ 1 & -1 & -1 & -\frac{2}{3} & -1 & -\frac{2}{3} & -\frac{2}{3} & -6 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2 & \frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & \frac{1}{3} & -\frac{4}{3} & -\frac{4}{3} & 2 \\ -0.5 & 1 & 1 & -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{1}{3} & 6 \end{bmatrix}.$$

Now for a chosen $P > 0$, i.e. Lyapunov function $V(x) = x^T Px$, by (12), we denote

$$\dot{V} = \left(x^s\right)^T \left(\tilde{P}\tilde{A}_i + \tilde{A}_i^T\tilde{P}\right)x^s, \quad s = 2,\ i = 1, 2. \tag{37}$$

As discussed before, we can search stable region of A_i over $P^1(\mathbb{R})$. It is done as follows: Let $x = (\cos(\theta), \sin(\theta))^T$, $\theta \in [-\pi/2, \pi/2]$. Then by (37) the region satisfies A_i, $i = 1, 2$,

$$a_i \tan^4(\theta) + b_i \tan^3(\theta) + c_i \tan^2(\theta) + d_i \tan(\theta) + e_i < 0, \tag{38}$$

where a_i, b_i, c_i, d_i, e_i, $i = 1, 2$, are constants related to P and A_i.

The stable region can be obtained by (38) easily.

Choosing $P = I$, then for the above A_1 the solution of (38) is

$$\theta \in U_A = \left(37.50955142166248°, 90°\right] \cup \left[-90°, -23.47330795373600°\right).$$

So the stable region of A_1 can be expressed in polar coordinate frame as

$$S_I(A_1) = \left\{(r, \theta) \mid r \in \mathbb{R},\ \theta \in U_A\right\}.$$

Similarly, for A_2 we have

$$\theta \in U_B = \left(-42.11583520926064°, 36.40524207176099°\right).$$

Then

$$S_I(A_2) = \{(r,\theta) \mid r \in \mathbb{R},\ \theta \in U_B\}.$$

Note that in $S_I(A_1)$, etc. we allow $r < 0$. A simple computation yields that both A_1 and A_2 are unstable.

Based the values of θ above, we find

$$S_I(A_1) \cup S_I(A_2) \subset \mathbb{R}^2.$$

This implies that system (33) is unstable under switching law determined by common Lyapunov function $x^T Px = \|x\|$.

However, if we choose the positive define

$$P = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

then we have

$$U_{A_1} = \big(0^\circ, 90^\circ\big] \cup \big[-90^\circ, -30.96375653207352^\circ\big),$$

and

$$U_{A_2} = \big(-38.56696195309441^\circ, 29.68630280257416^\circ\big).$$

Hence $S_P(A_1) \cup S_P(A_2) = \mathbb{R}^2$. According to Proposition 1 the system is stabilizable by state feedback switching law.

6. Conclusions

This paper investigated quadratic stability of switched homogeneous nonlinear systems. Firstly, using semi-tensor product of matrices, Lyapunov-like equation was obtained for stable homogeneous systems which was reduced to linear Lyapunov equation for linear systems. Based on this result, stability of switched homogeneous systems was studied. A switching law was designed by partition of state space, which was proved equivalent to convex combination of vector fields given by [14]. Especially, this designed switching law can avoid chattering phenomena which caused by switching rule. For planar switched homogeneous systems, we gave a sufficient condition for stability by LMI approach as in switched linear systems. As a result, it was easy to be realized in computer. At last, an illustrating example was given to explain importance of common Lyapunov function chosen.

Acknowledgments

The authors thank the anonymous reviewers and the Associate Editor for their valuable comments.

Appendix A. Semi-tensor product of matrices

Semi-tensor product of matrices is a new concept and a useful tool proposed through this paper, which is used to handle higher dimensional data, multi-linear mappings and polynomials.

A multi-variable polynomials can be expressed as a tensor form, i.e., a multi-linear form, then the semi-tensor product can be used.

We give a brief introduction for semi-tensor product. Details can be found in [27].

Definition 4.

1. Let X be a row vector of dimension np, Y be a column vector of dimension p. Express $X=(X^1,\dots,X^p)$, $X^i\in\mathbb{R}^n$, and define the left semi-tensor product, $\ltimes$, as

$$\begin{cases} X\ltimes Y=\displaystyle\sum_{i=1}^{p}X^i y_i\in\mathbb{R}^n,\\ Y^T\ltimes X^T=\displaystyle\sum_{i=1}^{p}y_i\left(X^i\right)^T\in\mathbb{R}^n. \end{cases} \tag{40}$$

2. Let $A\in M_{m\times n}$ and $B\in M_{p\times q}$. If either $nt=p$ or $n=pt$, define the left semi-tensor product of A and B, denoted by $C=A\ltimes B$, as $C=(C^{ij})$ and each block is

$$C^{ij}=A^i\ltimes B_j,\quad i=1,\dots,m,\ j=1,\dots,q,$$

where A^i is ith row of A and B_j is the jth column of B.

Example 3.

1. Let $X=[1\ 2\ 3\ -1]$ and $Y=\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$.

$$X\ltimes Y=[1\quad 2]\cdot 1+[3\quad -1]\cdot 2=[7\quad 0].$$

2. Let

$$A=\begin{bmatrix}1 & 2\\ -2 & -1\end{bmatrix},\qquad B=\begin{bmatrix}1&2&3\\2&3&2\\1&1&1\\1&2&0\end{bmatrix}.$$

Then

$$A\ltimes B=\begin{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} & \begin{bmatrix}4\\7\end{bmatrix} & \begin{bmatrix}5\\2\end{bmatrix}\\ \begin{bmatrix}-3\\-5\end{bmatrix} & \begin{bmatrix}-5\\-8\end{bmatrix} & \begin{bmatrix}-7\\-4\end{bmatrix}\end{bmatrix}=\begin{bmatrix}3&4&5\\4&7&2\\-3&-5&-7\\-5&-8&-4\end{bmatrix}.$$

Note that when $n=p$ the left semi-tensor product coincides the conventional matrix product. It has some fundamental properties.

Proposition 4. *The left semi-tensor product satisfies* (*as long as the related products are well defined*):

1. (*Distributive rule*)

$$\begin{aligned}&A\ltimes(\alpha B+\beta C)=\alpha A\ltimes B+\beta A\ltimes C;\\&(\alpha B+\beta C)\ltimes A=\alpha B\ltimes A+\beta C\ltimes A,\quad \alpha,\beta\in\mathbb{R}.\end{aligned} \tag{41}$$

2. (*Associative rule*)

$$A \ltimes (B \ltimes C) = (A \ltimes B) \ltimes C;$$
$$(B \ltimes C) \ltimes A = B \ltimes (C \ltimes A). \tag{42}$$

Proposition 5.

1. *Assume A and B are of the proper dimensions such that $A \ltimes B$ is well defined. Then*
$$(A \ltimes B)^T = B^T \ltimes A^T. \tag{43}$$
2. *In addition assume both A and B are invertible, then*
$$(A \ltimes B)^{-1} = B^{-1} \ltimes A^{-1}. \tag{44}$$

Proposition 6. *Let $A \in M_{p\times q}$ and $B \in M_{m\times n}$. If $q = km$, then*

$$A \ltimes B = A(B \otimes I_k). \tag{45}$$

If $kq = m$, then

$$A \ltimes B = (A \otimes I_k)B. \tag{46}$$

Proposition 7. *Assume $A \in M_{m\times n}$ is given*:

1. *Let $Z \in \mathbb{R}^t$ be a row vector. Then*
$$A \ltimes Z = Z \ltimes (I_t \otimes A). \tag{47}$$
2. *Let $Z \in \mathbb{R}^t$ be a column vector. Then*
$$Z \ltimes A = (I_t \otimes A) \ltimes Z. \tag{48}$$

About the differential of matrix of functions we have

Proposition 8.

$$D\left(x^{k+1}\right) = \Phi_k \ltimes x^k, \tag{49}$$

where

$$\Phi_k = \sum_{i=0}^{k} I_{n^s} \otimes W_{[n^{k-s},n]}. \tag{50}$$

Assume a matrix $A_{m\times n} = (a_{ij})$. Denote

$$V_r(A) = (a_{11}, \ldots, a_{1n}, \ldots, a_{m1}, \ldots, a_{mn})^T$$

and

$$V_c(A) = (a_{11}, \ldots, a_{m1}, \ldots, a_{1n}, \ldots, a_{mn})^T.$$

Clearly we have

$$V_c(A) = V_r\left(A^T\right), \qquad V_r(A) = V_c\left(A^T\right).$$

About the conventional matrix product is denoted by semi-tensor product, we have

Proposition 9. *Assume $B \in M_{m\times n}$, $X \in M_{n\times s}$, then*

$$BX = \mathcal{A}_s^r(B) \ltimes V_r(X), \tag{51}$$

where $\mathcal{A}_s^r = B[I_n \otimes V_r^T(I_s)]$,

$$V_r(BX) = A \ltimes V_r(B). \tag{52}$$

Proposition 10.

1. *Let $X \in \mathbb{R}^t$ be a row vector, A is a $t \times t$ matrix, then*
$$XA = V_r^T A \ltimes X^T. \tag{53}$$
2. *Let $x \in \mathbb{R}$ is a column vector, then*
$$\left(x^T\right)^k = V_r^T\left(I_n^k\right) \ltimes x^k, \qquad x^l \ltimes V_r^T(I_{n^k}) = \left(I_{n^l} \otimes V_r^T(I_{n^k})\right) \ltimes x^l.$$

Note that when $x \in \mathbb{R}$ is a column or a row, then $x \ltimes \cdots \ltimes x$ is well defined. Then we denote

$$x^k := \underbrace{x \ltimes \cdots \ltimes x}_{k}.$$

For later discuss, we need the tensor expression of polynomials. From Proposition 10, we can find x^k is a basis of kth degree polynomials. A kth degree homogeneous polynomial can be expressed as $\alpha \ltimes x^k$, where the coefficient vector α is a $1 \times n^k$ row, briefly, $\alpha \ltimes x^k := \alpha x^k$. Similarly, A vector filed of kth degree homogeneous polynomials can be expressed as $F \ltimes x^k$, where the coefficient vector F is an $n \times n^k$ matrix, briefly, $Fx^k := F \ltimes x^k$.

Proposition 11. *Let X and Y be kth and sth degree homogeneous polynomial vector fields. Then we can express X and Y as*

$$X = FX^k, \qquad Y = GY^s,$$

where F and G are $n \times n^k$ and $n \times n^s$ matrices, respectively. Then

$$X \ltimes Y = Fx^k \ltimes Gx^s = F \ltimes \left(x^k \ltimes G\right) \ltimes x^s = F(I_{n^k} \otimes G)x^{k+s}. \tag{54}$$

References

[1] D. Liberzon, A.S. Morse, Basic problems in stability and desing of switched systems, IEEE Control Syst. Mag. 19 (1999) 59–70.

[2] W. Chen, D.J. Balance, On a switching control scheme for nonlinear systems with ill-defined relative degree, Systems Control Lett. 47 (2002) 159–166.

[3] D. Cheng, Controllability of switched bilinear systems, IEEE Trans. Automat. Control 50 (4) (April 2005) 511–515.

[4] D. Cheng, L. Guo, J. Huang, On quadratic Lyapunov functions, IEEE Trans. Automat. Control 48 (5) (2003) 885–890.

[5] D. Cheng, Stabilization of planar switched linear systems, Systems Control Lett. 51 (2004) 79–88.

[6] Z. Sun, S.S. Ge, T.H. Lee, Controllability and reachability criteria for switched linear systems, Automatica 38 (2002) 775–786.

[7] X. Xu, P.J. Antsaklis, Optimal control of switched systems based on parameterization of the switching instants, IEEE Trans. Automat. Control 49 (1) (2004) 2–16.

[8] J.L. Mancilla-Aguilar, A condition for the stabilkity of switched nonlinear systems, IEEE Trans. Automat. Control 45 (11) (Nov. 2000).
[9] M. Margaliot, D. Liberzon, Lie-algebraic stability conditions for nonlinear switched systems and differential inclusions, Systems Control Lett., in press.
[10] W.P. Dayawansa, C.F. Martin, A converse Lyapunov theorem for a class of dynamical systems which undergo switching, IEEE Trans. Automat. Control 44 (1999) 751–764.
[11] J.L. Mancilla-Aguilar, R.A. Garcia, A converse Lyapunov theorem for nonlinear switched systems, Systems Control Lett. 41 (2000) 67–71.
[12] Z.G. Li, W.X. Xie, C.Y. Wen, Y.C. Soh, Globally exponential stabilization of switched nonlinear systems with arbitrary switchings, in: IEEE CDC, Sydney, Australia, 2000, pp. 3610–3615.
[13] W.X. Xie, C.Y. Wen, Z.G. Li, Input-to-state stabilization of switched nonlinear system, IEEE Trans. Automat. Control 46 (7) (July 2001) 1111–1116.
[14] M.A. Wicks, P. Peleties, R.A. Decarlo, Construction of piecewise Lyapunov functions for stabilizing switched systems, in: IEEE CDC, Lake Buena Vista, DE 1994, pp. 3492–3497.
[15] M.A. Wicks, P. Peleties, R.A. Decarlo, Switched controller synthesis for quadratic stabilization of a pair of unstable linear systems, Eur. J. Control 4 (2) (1998) 140–147.
[16] Z.G. Li, C.Y. Wen, Y.C. Soh, Stabilization of a class of switched systems via designing switching law, IEEE Trans. Automat. Control 46 (4) (2001) 665–670.
[17] D. Holcman, M. Margaliot, Stability analysis of second-order switched homogeneous systems, SIAM J. Control Optim. 41 (5) (2003) 1609–1625.
[18] H. Nakamura, Y. Yamashita, H. Nishitani, Smooth Lyapunov function for homogeneous differential inclusions, in: SICE, Osaka, August 2002, pp. 1974–1979.
[19] Y.G. Hong, H.Y. Li, Nonlinear H_∞ control and related problems of homogeneous systems, Internat. J. Control 71 (1) (1998) 79–92.
[20] Y.G. Hong, H_∞ control, stabilization, and input–output stability of nonlinear systems with homogeneous properties, Automatica 37 (6) (June 2001) 819–829.
[21] A. Andreeni, A. Baccciltti, G. Stefani, Global stabilizability of homogeneous vector fields of odd degree, Systems Control Lett. 10 (1988) 251–256.
[22] L. Zhang, D. Cheng, J. Liu, Stability of switched linear systems, Asian J. Control 5 (4) (2003) 476–483.
[23] L. Zhang, D. Cheng, C. Li, Disturbance decoupling of switched nonlinear systems, IEE Proc. Control Theory Appl. 152 (1) (2005) 49–54.
[24] L. Vu, D. Liberzon, Common Lyapunov functions for families of commuting nonlinear systems, Systems Control Lett. 4 (5) (May 2005) 405–416.
[25] D. Cheng, J. Ma, Calculation of stability region, in: IEEE CDC, vol. 6, Maui, Hawaii, USA, December 2003, pp. 5615–5620.
[26] D. Cheng, Semi-tensor product of matrices and its application to Morgen's problem, Chinese Sci. Ser. F 44 (3) (2001) 195–212.
[27] D. Cheng, Matrix and Polynomial Approach to Dynamic Control Systems, Science Press, Beijing, China, 2002.
[28] J.L. Kelley, General Topology, Springer-Verlag, 1955.
[29] Z.R. Xi, Y.G. Hong, D.Z. Cheng, Nonlinear H_∞ control of homogeneous systems via output feedback, J. Control Theory Appl. 17 (3) (June 2000) (in Chinese).
[30] Z. Ji, L. Wang, G. Xie, F. Hao, LMI approach to quadratic stabilization of switched systems, IEE Proc. Control Theory Appl. 151 (3) (May 2004) 289–294.
[31] D. Cheng, C. Martin, Stabilization of nonlinear systems via designed center manifold, IEEE Trans. Automat. Control 46 (9) (2001) 1372–1383.